Phase transition and Thermodynamical geometry of Reissner-Nordström-AdS Black Holes in Extended Phase Space

Jia-Lin Zhang[1,2], Rong-Gen Cai[1,2] and Hongwei Yu[1,3]

[1] *Department of Physics and Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha 410081, China*

[2] *State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

[3] *Center for Nonlinear Science and Department of Physics, Ningbo University, Ningbo 315211, China*

Abstract

We study the thermodynamics and thermodynamic geometry of a five-dimensional Reissner-Nordström-AdS black hole in the extended phase space by treating the cosmological constant as being related to the number of colors in the boundary gauge theory and its conjugate quantity as the associated chemical potential. It is found that the contribution of the charge of the black hole to the chemical potential is always positive and the existence of charge make the chemical potential become positive more easily. We calculate the scalar curvatures of the thermodynamical Weinhold metric, Ruppeiner metric and Quevedo metric, respectively, in the fixed N^2 case and the fixed q case. It is found that in the fixed N^2 case the divergence of the scalar curvature is related to the divergence of the specific heat with fixed electric potential in the Weinhold metric and Ruppeiner metric, and the divergence of the scalar curvature in the Quevedo metric corresponds to the divergence of the specific heat with fixed electric charge density. In the fixed q case, however, the divergence of the scalar curvature is related to the divergence of the specific heat with fixed chemical potential in the Weinhold metric and Ruppeiner metric, while in the Quevedo metric the divergence of the scalar curvature corresponds to the divergence of the specific heat with fixed number of colors and the vanishing of the specific heat with fixed chemical potential.

PACS numbers: 04.60.-m, 04.70.Dy, 11.25.-w

I. INTRODUCTION

The well-known AdS/CFT correspondence relates a weakly coupled gravitational theory in d-dimensional anti-de Sitter (AdS) spacetime to a strongly coupled conformal field theory (CFT) in a $(d-1)$-dimensional boundary of the AdS space [1–4] (for a review, see [5]). In the spirit of the AdS/CFT correspondence, the Hawking-Page phase transition between the stable large black hole and thermal gas in AdS space [6] can be interpreted as the confinement/deconfinement phase transition in the dual strongly coupled gauge theory [4]. Indeed in the past years we have witnessed increasing interest in studying thermodynamics and the phase structure of black holes in AdS space.

For a Kerr-Newmann black hole in general relativity, Davies [7] found that some heat capacities diverge at some values of black hole parameters, and he argued that some second order phase transitions will happen in the Kerr-Newmann black hole. For a Reissner-Nordström AdS (RN-AdS) black hole, such a phase transition was studied in some details in [8]. In a canonical ensemble with a fixed charge, in particular, it was found that there exists a phase transition between small and large black holes. This phase transition behaves very like the gas/liquid phase transition in a Van der Waals system [8, 9]. However, an complete identification between a RN-AdS black hole and the Van der Waals system was recently realized in [10], where the negative cosmological constant plays the role as pressure, while its conjugate acts as the thermodynamic volume of the black hole in the so-called extended phase space [11, 12] (For a recent review, see [13]). Recently, there has been a lot of works studying the thermodynamics and phase transition in the extended phase space for black holes in AdS space (For an incomplete list of reference see [14]).

In the AdS/CFT correspondence, the negative cosmological constant is related to the degrees of freedom of the dual CFT. Thus an interesting question arises as to whether the interpretation of the cosmological constant as pressure is applicable to the boundary CFT. Very recently, it was argued that it is more suitable to view the cosmological constant as the number of colors in the dual gauge field and its conjugate as associated chemical potential [15–17]. This interpretation was examined for $\mathcal{N}$=4 supersymmetric Yang-Mills theory at large N in [15, 18], by studying the corresponding thermodynamics of a Schwarzschild-AdS black hole. They calculated the chemical potential conjugate to the number of colors, and found that the chemical potential in the high temperature phase of the Yang-Mills theory

is negative and decreases as temperature increases. For spherical black holes in the bulk the chemical potential approaches zero as the temperature is lowered below the Hawking-Page temperature and changes its sign at a temperature near the temperature at which the heat capacity diverges. This phenomenon might be related to the Bose-Einstein condensation in the dual field theory [15]. Furthermore, in [17] the authors studied the associated chemical potential from the point of view of holographic entanglement entropy.

On the other hand, recently a lot of attention has been attracted to applying the thermodynamical geometry to thermodynamics and phase transition of black holes. The geometrical ideas to ordinary thermodynamical systems were first introduced by Weinhold [19]. He considered a kind of metric defined as the second derivatives of internal energy with respect to entropy and other extensive quantities for a thermodynamic system. Based on the fluctuation theory of equilibrium thermodynamics, Ruppeiner [20] introduced another metric defined as the minus second derivatives of entropy with respect to the internal energy and other extensive quantities. He argued that the scalar curvature of the Ruppeiner metric can reveal the micro interaction of the system and its divergence is related to a certain phase transition [21]. In fact the Weinhold metric is conformal to the Ruppeiner metric with the inverse temperature as the conformal factor [22]. Unfortunately, both the Weinhold metric and Ruppeiner metric are not invariant under Legendre transformation. A few years ago Quevedo *et al.* [23–26] proposed an approach to obtain a new metric which is Legendre invariant in the space of equilibrium state. As far as we know, applying the thermodynamical geometry to black hole thermodynamics started in [27], which shows that the Weinhold metric is proportional to the metric on the moduli space for supersymmetric extremal black holes with vanishing Hawking temperature, and the Ruppeiner metric governing fluctuations naively diverges, which is consistent with the fact that near the extremal limit, the thermodynamical description of black holes should be invalid. Applying the thermodynamical geometry approach to the phase transition of black holes was followed in [28, 29], and for more recent references see the review paper [30] and references therein. In particular, Ref. [31] has investigated the relation between the divergence of the scalar curvature of thermodynamical geometry in different ensembles and the singularity of heat capacities. In a previous paper we have studied thermodynamical geometry and the phase transition for a Schwarzschild AdS black hole in the extended phase space where the cosmological constant is related to the number of colors of dual gauge field [18].

In this paper, we will extend the previous study to the case of a charged black hole in AdS space, and concretely we will study the thermodynamics and thermodynamical geometry for a five-dimensional RN-AdS black hole by viewing the number of colors as a thermodynamical variable from the viewpoint of dual CFT. For this, we will calculate energy density and entropy density for the dual CFT and then obtain the chemical potential associated with the number of colors in the next section. In Sec. III and IV, we will calculate the thermodynamical curvatures of the Weinhold metric, Ruppeiner metric and Quevedo metric, respectively, for the thermodynamical system in the fixed N^2 case and the fixed q case, in order to see the relation between the thermodynamical curvature and the phase transition of black holes in AdS space. We end the paper with conclusions in Sec. V.

II. THERMODYNAMICS OF RN-ADS BLACK HOLES IN EXTENDED PHASE SPACE

Let us start with the following Einstein-Maxwell theory with a negative cosmological constant in five-dimensional spacetime

$$S = \frac{1}{16\pi G_5}\int_M d^5x\sqrt{-g}\left[R - L^2F^2 + \frac{12}{L^2}\right], \tag{1}$$

where L is the AdS radius related to the cosmological constant as $\Lambda = -6/L^2$. As shown in [8], the above action with an additional Chern-Simons term can be viewed as an effective truncation of type IIB supergravity on an S^5. The action admits a five-dimensional RN-AdS black hole as its exact solution, which can be written in the static coordinates as

$$ds_5^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij}dx^i dx^j, \tag{2}$$

where $h_{ij}dx^i dx^j$ is the line element of a three-dimensional Einstein space Σ_3 with constant curvature $6k$, and the metric function $f(r)$ is given by [8]

$$f(r) = k - \frac{m}{r^2} + \frac{r^2}{L^2} + \frac{q_L^2}{r^4}\ . \tag{3}$$

The integration constant m is related to the mass of the black hole

$$M = \frac{3\omega_3}{16\pi G_5}m \tag{4}$$

where ω_3 denotes the volume of Σ_3, while the parameter q_L has a relation to the physical charge Q of the black hole as

$$q_L = \frac{4\pi G_5 Q L}{\sqrt{3}\omega_3} \ . \tag{5}$$

Without loss of generality, one can take the scalar curvature parameter k of the three-dimensional space Σ_3 as $k = 1$, 0, or -1, respectively. Uplifting this solution (2) to ten dimensions, one has [8]

$$ds_{10}^2 = ds_5^2 + L^2 \sum_{i=1}^{3} [d\mu_i^2 + \mu_i^2 (d\varphi_i + \frac{2}{\sqrt{3}} A_\mu dx^\mu)^2], \tag{6}$$

where $\mu = 0, \cdots, 4$, the variables μ_i are direction cosines on S^5 satisfying $\sum_{i=1}^3 \mu_i^2 = 1$, and the φ_i are rotation angles on S^5. The ten-dimensional spacetime (6) can be viewed as the near horizon geometry of N rotating black $D3$-branes in type IIB supergravity. In that case, the AdS radius L has a relation to the number N of D3-branes [1]

$$L^4 = \frac{\sqrt{2} N \ell_p^4}{\pi^2} \equiv \alpha^2 N, \tag{7}$$

where ℓ_p is the ten-dimensional Planck length. According to the AdS/CFT correspondence, the spacetime (6) can be regarded as the gravity dual to $\mathcal{N}$=4 supersymmetric Yang-Mills theory in the Coulomb branch. Then N is nothing, but the rank of the gauge group of the supersymmetric $SU(N)$ Yang-Mills Theory. In the large N limit, the number of degrees of freedom of the $\mathcal{N}$=4 supersymmetric Yang-Mills theory is proportional to N^2 [32].

The black hole horizon r_h is determined by equation $f(r) = 0$ by taking the largest real root of the equation. Then with Eq. (3) and Eq. (4), the mass of black hole can be expressed as

$$M = \frac{3\omega_3}{16\pi G_5} \big(k r_h^2 + \frac{r_h^4}{L^2} + \frac{4 G_5^2 Q^2 L^2}{3\pi^2 r_h^2} \big) \ . \tag{8}$$

Using the Bekenstein-Hawking entropy formula of black hole, we have the black hole entropy

$$S = \frac{A}{4G_5} = \frac{\omega_3 r_h^3}{4G_5} \ . \tag{9}$$

Note that $G_5 = G_{10}/(\pi^3 L^5)$ and $G_{10} = \ell_p^8$. Furthermore, let us notice that the dual CFT to the RN-AdS black hole lives in the AdS boundary with a metric (up to a conformal factor)

$$ds^2 = -dt^2 + L^2 h_{ij} dx^i dx^j. \tag{10}$$

Namely the CFT lives in a space with volume $V_3 = \omega_3 L^3$. We see that the volume depends on the number of colors N^2. In order to remove the effect of volume change when one varies

the number of colors, let us consider the corresponding densities of some thermodynamic quantities of the dual CFT as

$$\rho = \frac{M}{V_3} = \frac{3\pi^2 L^2}{16G_{10}} r_h^2 (k + \frac{r_h^2}{L^2}) + \frac{G_{10}Q^2}{4L^6\pi^6 r_h^2}, \tag{11}$$

$$s = \frac{S}{V_3} = \frac{\pi^3}{4G_{10}} L^2 r_h^3, \tag{12}$$

$$q = \frac{Q}{V_3} = \frac{Q}{2\pi^2 L^3}\ . \tag{13}$$

The Hawking temperature of the black hole can be given by requiring the absence of the potential conical singularity of the Euclidean black hole at the horizon. A simple calculation gives

$$T = \frac{1}{2\pi r_h}\left(k + 2\frac{r_h^2}{L^2}\right) - \frac{2G_{10}^2 Q^2}{3L^8\pi^9 r_h^5}. \tag{14}$$

The energy density can be expressed in terms of the entropy density, the number of colors and the charge density as

$$\rho = \frac{3D\pi^2}{16G_{10}}\left[\alpha k N^{1/6} s^{2/3} + N^{-2/3} s^{4/3} D\right] + \frac{D_1}{2} N^{1/3} q^2 s^{-2/3}\ , \tag{15}$$

where we have introduced $D \equiv \left[4G_{10}/(\pi^3\alpha)\right]^{2/3}$ and $D_1 \equiv \left(G_{10}\alpha^2/2\right)^{1/3}$ for convenience in the following discussions. According to the standard thermodynamic relations, the corresponding intensive variables of the CFT can be calculated. For example, the temperature of the CFT is

$$T = \left(\frac{\partial \rho}{\partial s}\right)_{N^2,q} = \frac{D\pi^2}{8G_{10}}\left(\alpha k N^{1/6} s^{-1/3} + 2N^{-2/3} s^{1/3} D\right) - \frac{D_1}{3} N^{1/3} q^2 s^{-5/3}, \tag{16}$$

which is noting, but just the Hawking temperature Eq. (14) of the black hole. In order to keep that the RN-AdS black hole describes a thermal state of the dual field theory, the Hawking temperature must be non-negative, i.e., the following condition has to be satisfied

$$3D\pi^2 s^{4/3}\left(2Ds^{2/3} + kN^{5/6}\alpha\right) \geq 8D_1 G_{10} N q^2\ . \tag{17}$$

Here the equality means an extremal black hole with vanishing temperature $T = 0$. The static electric potential associated with the charge of the black hole can also be obtained as

$$\Phi = \left(\frac{\partial \rho}{\partial q}\right)_{s,N^2} = D_1 q N^{1/3} s^{-2/3}\ . \tag{18}$$

This can be explained by the chemical potential associated with R current in the dual Supersymmetric Yang-Mills theory. But we remind the readers not to be confused with the

following chemical potential associated with the number of colors in the dual field theory. The chemical potential μ conjugate to the number of colors is defined as [15]

$$\mu = \left(\frac{\partial \rho}{\partial N^2}\right)_{s,q} = \frac{D\pi^2}{16G_{10}}\left(\frac{1}{4}\alpha k N^{-11/6} s^{2/3} - N^{-8/3} s^{4/3} D\right) + \frac{D_1}{12} q^2 N^{-5/3} s^{-2/3}, \tag{19}$$

which is the measure of the energy cost to the system of increasing the number of colors. We see that the contribution of electric charge to the chemical potential is always positive.

As a result, we have the first law of thermodynamics

$$d\rho = T ds + \mu dN^2 + \Phi dq. \tag{20}$$

Clearly one can see that the case relating the cosmological constant to the number of colors is quite different from the case viewing the cosmological constant as the pressure. In the former case, one can see from (20) that the mass of the black hole can be viewed as the internal energy of the system, while in the latter case, the mass of the black hole is taken as the enthalpy [11, 12].

For the cases of $k = 0$ and $k = -1$, it is easy to see that from Eq. (16) for fixed N^2 and q, the Hawking temperature increases monotonically with the entropy density s. Besides, with the inequality (17) for the case of $k = 0$ or $k = -1$, it is not difficult to deduce from Eq. (19) that the chemical potential is always negative for non-extremal black holes. This is consistent with our understanding that the chemical potential for classical gas is negative and will become more negative as temperature increases [33]. Therefore, the cases of $k = 0$ and $k = -1$ are trivial for studying the phase transition and thermodynamic geometry. We will focus our attention on the case of $k = 1$ in what follows.

For the case of $k = 1$, we have

$$\frac{\partial T}{\partial s} = \frac{3D\pi^2 s^{4/3}\left(2Ds^{2/3} - N^{5/6}\alpha\right) + 40Nq^2 D_1 G_{10}}{72N^{2/3} s^{8/3} G_{10}} \,. \tag{21}$$

Then we can conclude that the Hawking temperature may not be a monotonic function of s by choosing other proper thermodynamic variables. For a fixed number of colors, there is a critical point of q, which can be determined by

$$\left(\frac{\partial T}{\partial s}\right)_{q_{crit}} = \left(\frac{\partial^2 T}{\partial s^2}\right)_{q_{crit}} = 0 \,. \tag{22}$$

It is easy to obtain that the critical point satisfies

$$q_{crit}^2 = \frac{N^{3/2}\pi^2\alpha^3}{360DD_1G_{10}} \,. \tag{23}$$

When $q > q_{crit}$, the Hawking temperature is always a monotonically increasing function of s . Similarly, if we fix the charge density q, there must be a critical point for the number of colors. For the case of $N < N_{crit} = \frac{12}{\alpha^2}\left(\frac{75D^2q^4D_1^2G_{10}^2}{\pi^4}\right)^{1/3}$, the Hawking temperature is always a monotonically increasing function of s , and otherwise, there must exist a certain interval of s in which the Hawking temperature is a monotonically decreasing function. In Fig. (1) we show the behavior of the temperature in the cases of $k = -1$, 0 and 1, respectively. We can see that in the case of $k = 1$, the temperature in the interval $(1.167, 3.443)$ is a monotonically decreasing function of s with parameters $N = 10$, $q = 0.4$ and $\ell_p = 1$.

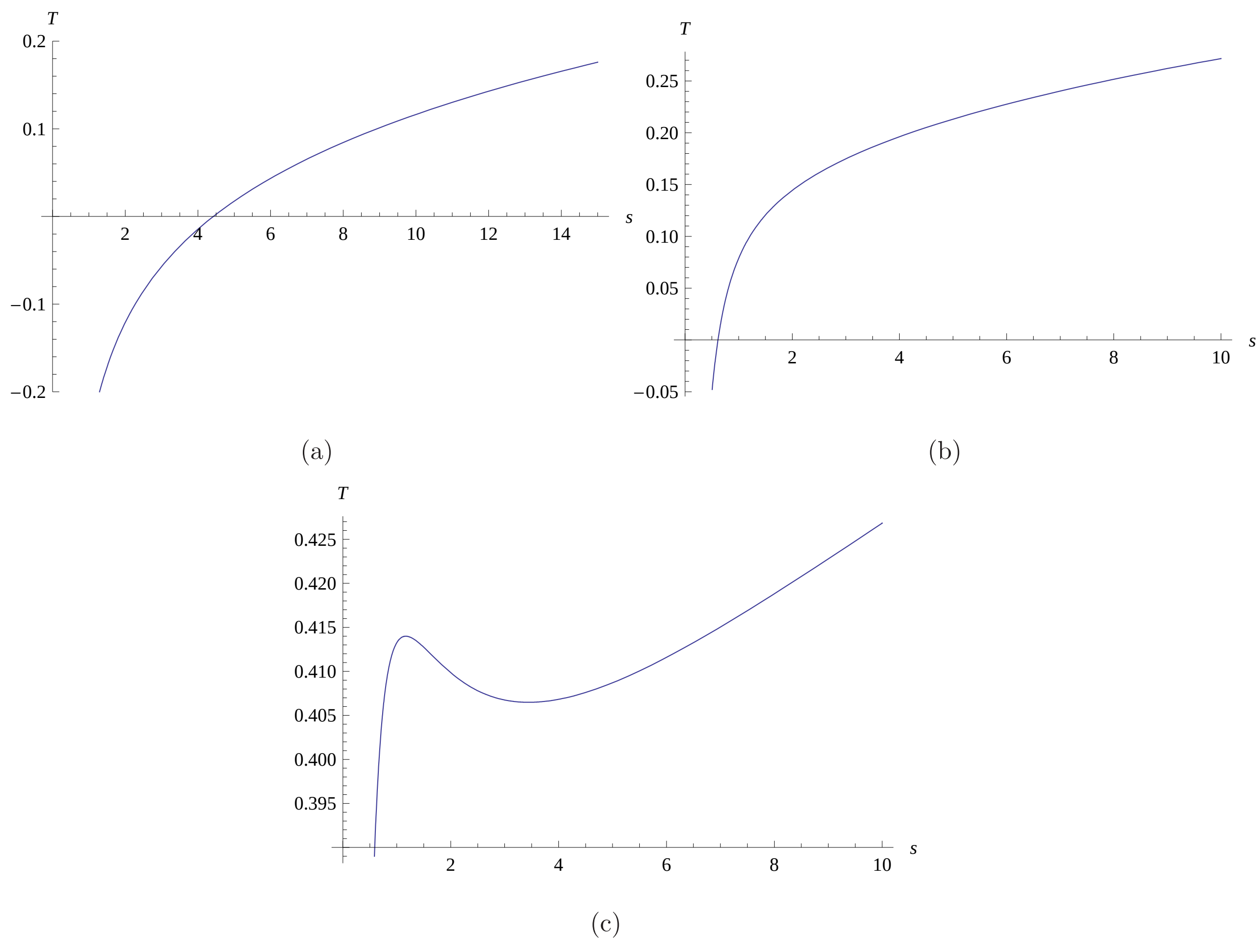


FIG. 1: The Hawking temperature vs entropy density s for (a)$k = -1$ (b)$k = 0$ (c)$k = 1$ with $N = 10$, $q = 0.4$, and $\ell_p = 1$.

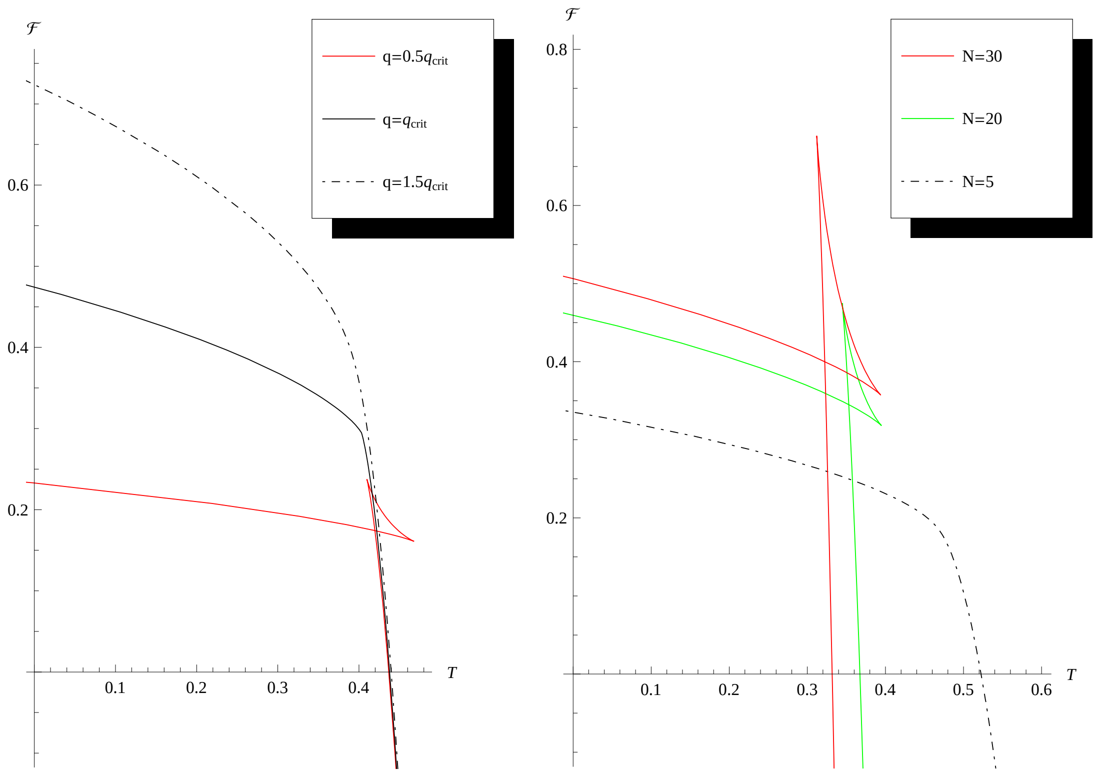
The Helmholtz free energy density can be calculated as

$$\mathcal{F} = \rho - Ts = \frac{5N^{1/3}q^2D_1}{6s^{2/3}} - \frac{D^2\pi^2s^{4/3}}{16N^{2/3}G_{10}} + \frac{DN^{1/6}\pi^2s^{2/3}\alpha}{16G_{10}} \,. \tag{24}$$

Note that the free energy also has a chance to be positive, if the following condition is satisfied

$$\frac{3D\pi^2 s^{4/3}\left(2Ds^{2/3}+N^{5/6}\alpha\right)}{8ND_1G_{10}} \geq q^2 > \frac{3D\pi^2 s^{4/3}\left(Ds^{2/3}-N^{5/6}\alpha\right)}{40ND_1G_{10}} \ . \tag{25}$$

The sign change of the free energy indicates the appearance of the Hawking-Page phase transition. In Fig. (2), we plot the free energy with respect to the Hawking temperature. It



(a)$N = 10$ (b)$q = 0.4$

FIG. 2: The Helmholtz free energy as a function of Hawking temperature for (a) a fixed number of colors and (b) a fixed charge density with $k = 1$ and $\ell_p = 1$. The "swallow tail" appears in figure (a) for $q < q_{crit} \approx 0.4817$ and in figure (b) for $N > N_{crit} = \frac{12}{\alpha^2}\left(\frac{75D^2q^4D_1^2G_{10}^2}{\pi^4}\right)^{1/3} \approx 7.805$.

can be seen that the "swallow tail", a type signal for the first order phase transition, appears in both the cases with a fixed number of colors and a fixed charge. The phase transition between the small black hole and the large one in the former case has been studied in [8], while the latter case is the new finding in this paper.

Now we turn to the case of the chemical potential conjugate to the number of colors. According to Eq. (19) and inequality (17), the condition for a positive chemical potential

can be written as

$$\frac{3D\pi^2 s^{4/3}\left(2Ds^{2/3}+N^{5/6}\alpha\right)}{8ND_1G_{10}} \geq q^2 > \frac{3D\pi^2 s^{4/3}\left(4Ds^{2/3}-N^{5/6}\alpha\right)}{16ND_1G_{10}} \quad . \tag{26}$$

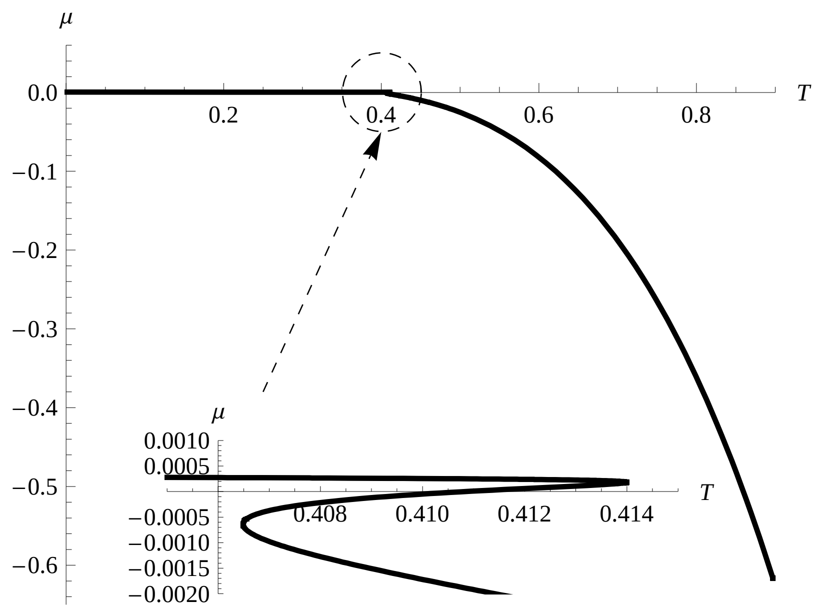


FIG. 3: The chemical potential vs the Hawking temperature with parameters $k = 1$, $N = 10$, $q = 0.4$ and $\ell_p = 1$.

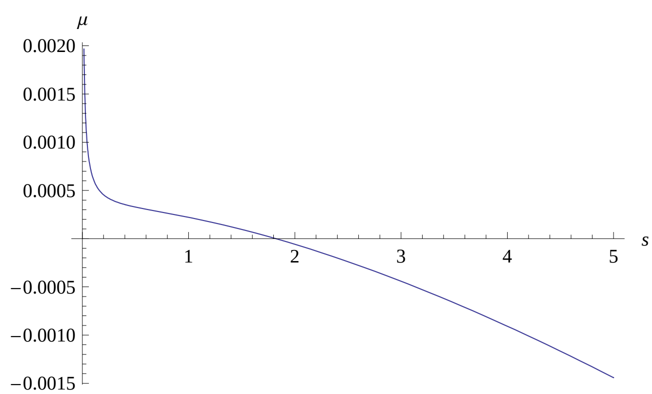


FIG. 4: The chemical potential as a function of s. Here we take $N = 10$, $q = 0.4$, $k = 1$ and $\ell_p = 1$.

In Fig. (3), we show the chemical potential as a function of temperature T for fixed q and N , while in Fig. (4) the chemical potential is plotted as a function of s for fixed $N = 10$ and $q = 0.4$. As we can see that the chemical potential surely has a chance to be positive by

choosing proper thermodynamic variables. It should be pointed out that when the chemical potential approaches zero and becomes positive, quantum effects should play some role [15]. The existence of the electric charge of black hole makes the chemical potential go beyond zero more easily on some level. We can see from Fig. (3) that there exists a multi-valued region, which just corresponds to the unstable region of the black hole with a negative heat capacity (see plot (c) in Fig. (1)), while Fig. (4) shows that the chemical potential is positive for small black holes.

In the following sections, we will study the phase transition and thermodynamical geometry of the RN-AdS black hole in the fixed N^2 case and the fixed q case, respectively.

III. PHASE TRANSITION AND THERMODYNAMIC GEOMETRY OF THE RN-ADS BLACK HOLE IN THE FIXED N^2 CASE

In this section, we will study the phase transition and thermodynamic geometry of the black hole in the ensemble with a fixed number of the colors. Once the number of the colors is fixed, it means that the cosmological constant is kept at a certain value and not treated as a thermodynamic variable. The corresponding phase transition and thermodynamics of RN-AdS black holes have been studied extensively in Refs. [8, 9, 34]. Here we pay attention to the relation between the phase transition and thermodynamic geometry. In this case, the associated specific heats can be calculated as

$$C_{q,N^2} = T\left(\frac{\partial s}{\partial T}\right)_{q,N^2} = \frac{3s\left[3D\pi^2 s^{4/3}\left(2Ds^{2/3} + N^{5/6}\alpha\right) - 8Nq^2 D_1 G_{10}\right]}{3D\pi^2 s^{4/3}\left(2Ds^{2/3} - N^{5/6}\alpha\right) + 40Nq^2 D_1 G_{10}}\ , \tag{27}$$

and

$$C_{\Phi,N^2} = T\left(\frac{\partial s}{\partial T}\right)_{\Phi,N^2} = \frac{3s\left[3D\pi^2 s^{4/3}\left(2Ds^{2/3} + N^{5/6}\alpha\right) - 8Nq^2 D_1 G_{10}\right]}{3D\pi^2 s^{4/3}\left(2Ds^{2/3} - N^{5/6}\alpha\right) + 8Nq^2 D_1 G_{10}}\ . \tag{28}$$

In the canonical ensemble with a fixed N^2 , a critical point exists and it can be calculated by Eq. (23). The behavior of the C_{Φ,N^2} as a function of s is plotted in Fig. (5) and the specific heat C_{q,N^2} is shown in Fig. (6). The behavior of $C_{q,N}$ is consistent with the one of the temperature shown in Fig. (1)(c): for small and large black holes the specific heat is positive, while it is negative for the intermediate black holes when q is less than the critical value, while the specific heat is always positive in the case when the charge q is larger than the critical one.

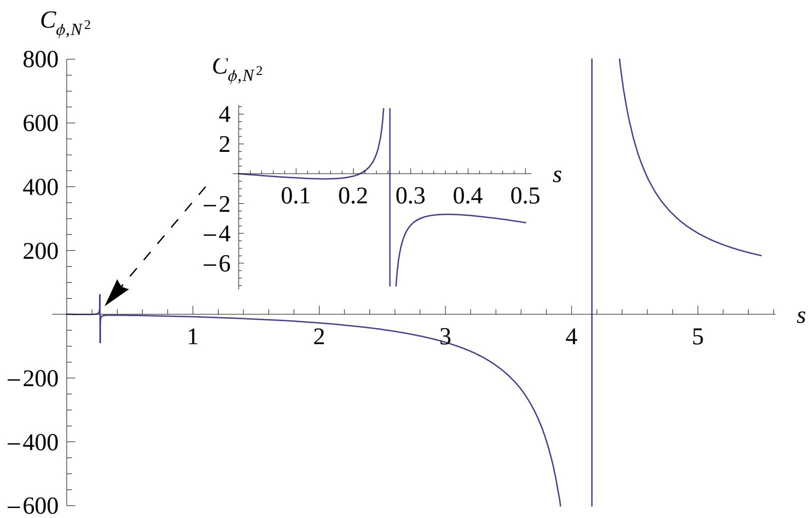


FIG. 5: The specific heat C_{Φ,N^2} of the RN-AdS black hole with respect to entropy density s for parameters $k = 1$, $N = 10$, $q = 0.4$ and $\ell_p = 1$. There are two divergent points at $s_1 \approx 0.264$ and $s_2 \approx 4.160$ in this case. It should be pointed out that $s < 0.211$ corresponds the situation of negative Hawking temperature, this nonphysical situation is not our main attention and not considered detailed in following paragraphs and figures.

Now we turn to the thermodynamical geometry of the black hole to see whether the thermodynamical curvature can reveal the singularity of these two specific heats. The Weinhold metric [19] is defined as the second derivatives of internal energy with respect to entropy and other extensive quantities in the energy representation, while the Ruppeiner metric [20] is related to the Weinhold metric by a conformal factor of temperature [22]

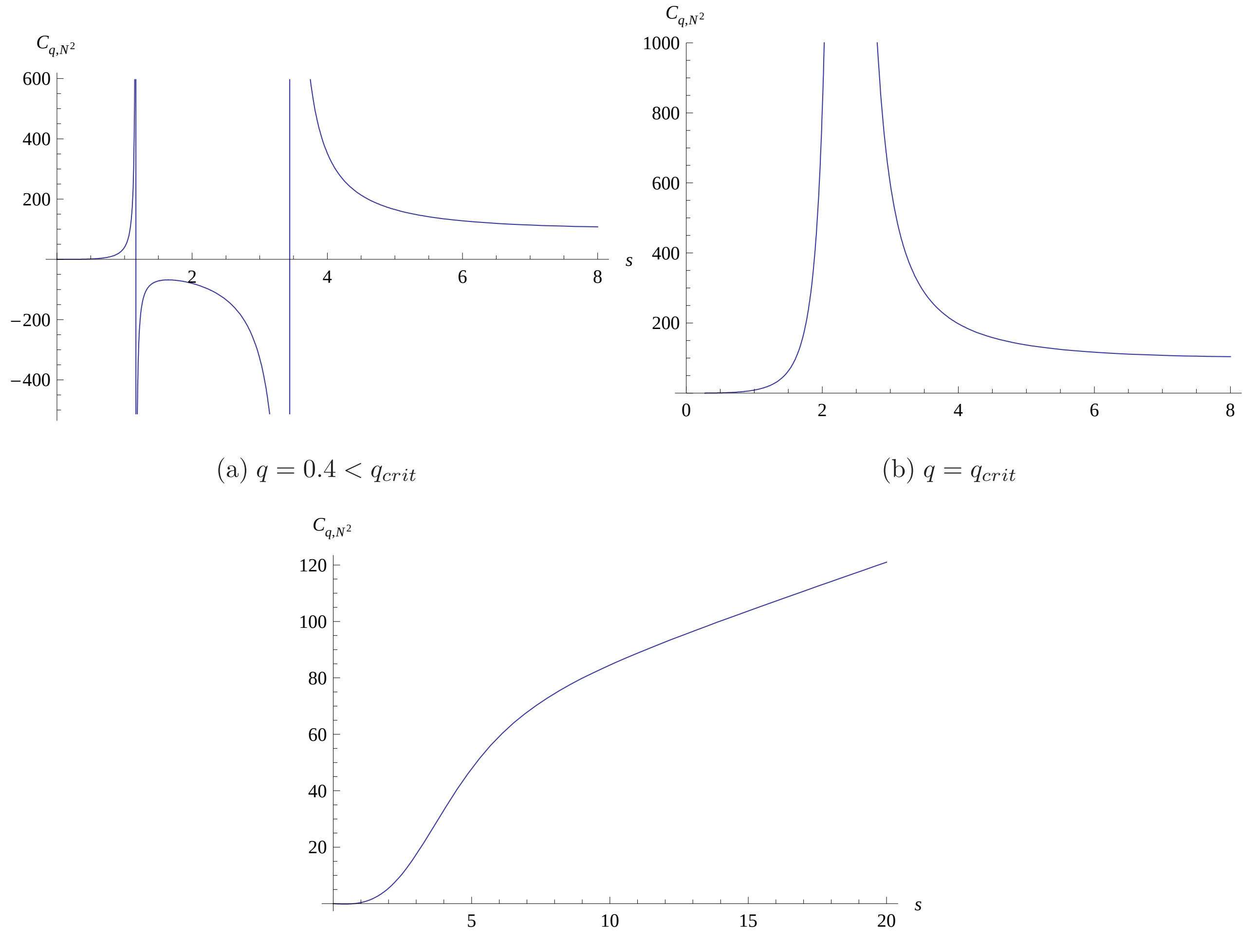
$$ds_R^2 = \frac{1}{T} ds_W^2 \ . \tag{29}$$

The Weinhold metric and Ruppeiner metric, which are dependent on the choice of thermodynamic potentials, are not Legendre invariant, while the Legendre invariant Quevedo metric is defined as [23–26]

$$g = \left(E^c \frac{\partial \phi}{\partial E^c}\right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \phi}{\partial E^c \partial E^d} dE^a dE^d\right) \ , \ \eta_{cd} = \mathrm{diag}(-1, 1, \dots, 1) \tag{30}$$

where ϕ denotes the thermodynamic potential, E^a and I^a respectively represent the set of extensive variables and the set of intensive variables, and $a = 1, 2, \cdots, n$.

Now we calculate the thermodynamical curvature for the RN-AdS black hole. The Wein-



(a) $q = 0.4 < q_{crit}$

(b) $q = q_{crit}$

(c) $q = 1 > q_{crit}$

FIG. 6: The specific heat C_{q,N^2} of the RN-AdS black hole with respect to entropy density for parameters $k = 1$, $N = 10$ and $\ell_p = 1$. (a): Two divergences at $s_3 \approx 1.167$ and $s_4 \approx 3.443$,(b): One divergence at $s_5 \approx 2.339$,(c) : No divergence.

hold metric is given by

$$g^W = \begin{pmatrix} \rho_{ss} & \rho_{sq} \\ \rho_{qs} & \rho_{qq} \end{pmatrix} , \tag{31}$$

where ρ_{ij} stands for $\partial^2\rho/\partial x^i \partial x^j$, and $x^1 = s$, $x^2 = q$. The scalar curvature of this metric can be calculated directly. Substituting Eq. (15) into Eq. (31) leads to the scalar curvature:

$$R^W = -\frac{48DN^{3/2}\pi^2 s^2 \alpha G_{10}}{\left[3D\pi^2 s^{4/3}\left(2Ds^{2/3} - N^{5/6}\alpha\right) + 8Nq^2 D_1 G_{10}\right]^2} . \tag{32}$$

On the other hand, by considering Eq. (29), the Ruppeiner metric can be written as

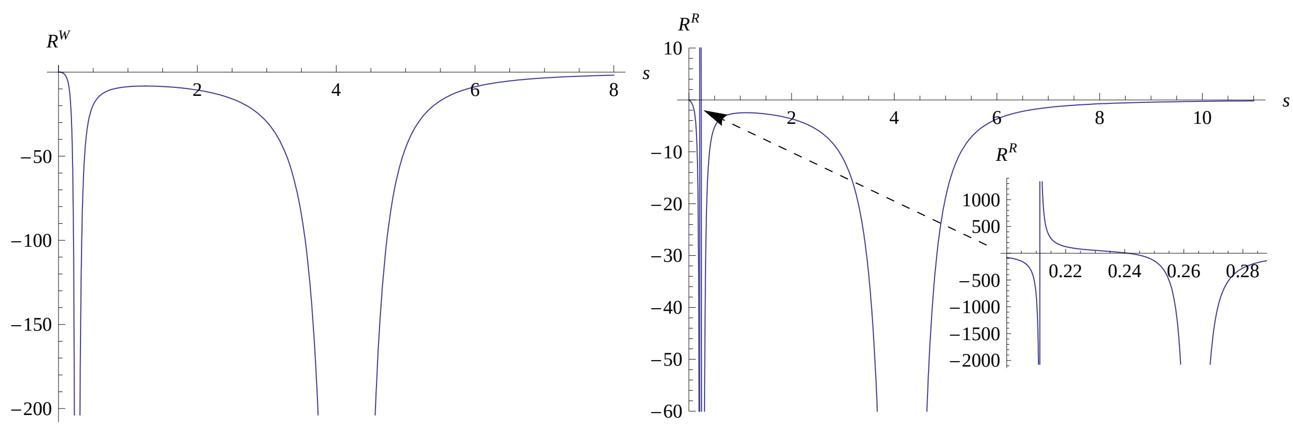
$$g^R = \frac{1}{T}\begin{pmatrix} \rho_{ss} & \rho_{sq} \\ \rho_{qs} & \rho_{qq} \end{pmatrix} , \tag{33}$$

and the corresponding curvature of this metric is

$$R^R = \frac{A_1(s,q)}{B_1(s,q)E_1(s,q)} , \tag{34}$$

where $A_1(s,q)$ is a regular function without any singular behavior, and the other two auxiliary functions are given by

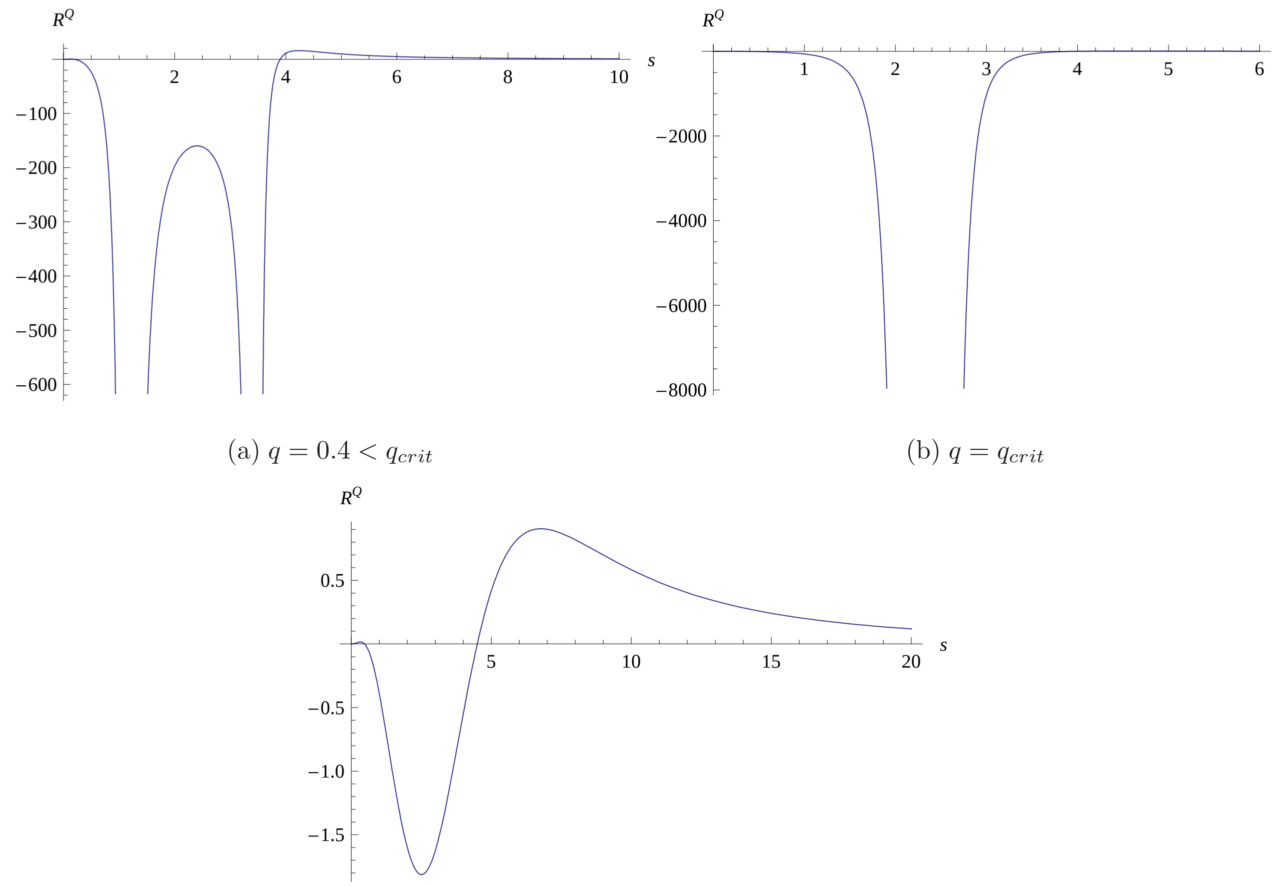
$$\begin{aligned} B_1(s,q) &= \left[3D\pi^2 s^{4/3}\left(2Ds^{2/3} - N^{5/6}\alpha\right) + 8G_{10}Nq^2D_1\right]^2 \\ E_1(s,q) &= 3D\pi^2 s^{4/3}\left(2Ds^{2/3} + N^{5/6}\alpha\right) - 8Nq^2D_1G_{10} . \end{aligned}$$



(a)Weinhold metric (b)Ruppeiner metric

FIG. 7: The scalar curvature of thermodynamic geometry vs entropy density with $k = 1$, $N = 10$, $q = 0.4$ and $\ell_p = 1$. Both scalar curvatures diverge at $s_1 \approx 0.264$ and $s_2 \approx 4.160$. Note that the third singularity at $s \approx 0.211$ for the scalar curvature of Ruppeiner metric corresponds to the extremal black hole with a vanishing Hawking temperature.

With the requirement for a positive Hawking temperature (17), it is easy to see that the auxiliary function $E_1(s,q)$ is always positive. Thus the singularities of R^R are just determined by the function $B_1(s,q)$. Therefore, from Eq. (32) and Eq. (34), we can conclude that both the scalar curvatures of the Weinhold metric and Ruppeiner metric possess the same singularities. These singularities just coincide with the divergence of the specific heat C_{Φ,N^2} for fixed electric potential and numbers of colors (comparing Fig. (5) with Fig. (7)).



(a) $q = 0.4 < q_{crit}$ (b) $q = q_{crit}$

(c) $q = 1 > q_{crit}$

FIG. 8: The scalar curvature of Quevedo metric vs entropy density with $k = 1$, $N = 10$ and $\ell_p = 1$ for various q . The scalar curvature diverges at (a) $s_3 \approx 1.167$ and $s_4 \approx 3.443$ for $q = 0.4$, (b) $s_5 \approx 2.339$ for $q = q_{crit}$, (c) no divergence for $q = 1 > q_{crit}$.

So both the Weinhold metric and Ruppeiner metric can reveal the phase transition of the RN-AdS black hole in the fixed Φ ensemble.

The Quevedo metric for the RN-AdS black hole reads

$$g^Q = (sT + q\Phi)\begin{pmatrix} -\rho_{ss} & 0 \\ 0 & \rho_{qq} \end{pmatrix} . \tag{35}$$

Calculating its scalar curvature gives

$$R^Q = \frac{A_2(s,q)}{B_2(s,q)E_2(s,q)}, \tag{36}$$

where $A_2(s,q)$ is a complicated regular function which we do not present here. The other

two auxiliary functions are given by

$$
\begin{aligned}
B_2(s,q) &= \left[3D\pi^2 s^{4/3}\left(2Ds^{2/3}-N^{5/6}\alpha\right)+40Nq^2D_1G_{10}\right]^2, \\
E_2(s,q) &= \left(6D^2\pi^2s^2+3DN^{5/6}\pi^2s^{4/3}\alpha+16Nq^2D_1G_{10}\right)^3 .
\end{aligned}
$$

It is easy to see that $E_2(s,q)$ is a positive function and the divergence of the scalar curvature for the Quevedo metric just corresponds to the singular points of the specific heat C_{q,N^2} by comparing the forms of E_2 and C_{q,N^2} (see Fig. (6) and Fig. (8)). This means that the Quevedo metric can reveal the phase transition of the RN-AdS black hole in the fixed q ensemble.

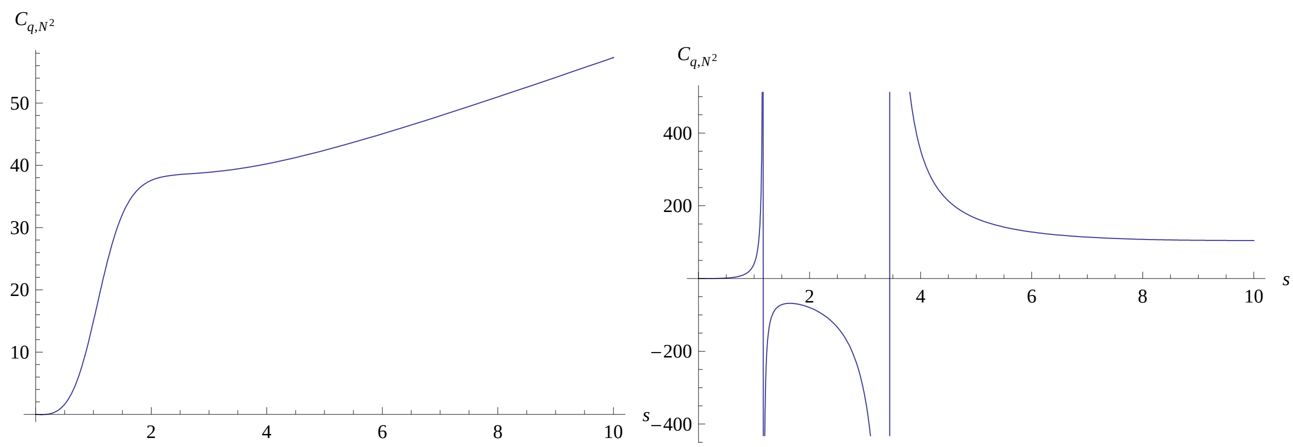
IV. PHASE TRANSITION AND THERMODYNAMIC GEOMETRY OF THE RN-ADS BLACK HOLE IN THE FIXED q CASE

Now we turn to the fixed q case. This means that the charge density q is treated as a fixed external parameter, not a thermodynamic variable. Then the corresponding specific heats are $C_{N^2,q}$ and $C_{\mu,q}$. The exact expression of $C_{N^2,q}$ is given in Eq. (27) in the previous section, from which we can see there is a critical value N_{crit}, for the number of colors. When $N < N_{crit} = \frac{12}{\alpha^2}\left(\frac{75D^2q^4D_1^2G_{10}^2}{\pi^4}\right)^{1/3}$, there does not exist any divergence for the specific heat $C_{N^2,q}$. On the other hand, when $N > N_{crit}$, there exist some divergences for this specific heat, which indicates some phase transition in canonical ensemble with a fixed q. We plot the behavior of $C_{N^2,q}$ as a function of s in Fig. (9) with fixed $q = 0.4$ for two values of the number of colors to show the feature. On the other hand, the specific heat $C_{\mu,q}$ with a fixed chemical potential can be calculated as

$$
C_{\mu,q} = T\left(\frac{\partial s}{\partial T}\right)_{\mu,q} = \frac{C_1(s,N)C_2(s,N)}{C_3(s,N)} , \tag{37}
$$

where the auxiliary functions read

$$
\begin{aligned}
C_1(s,N) &= 3D\pi^2s^{4/3}\left(64Ds^{2/3}-11N^{5/6}\alpha\right)-160Nq^2D_1G_{10} , \\
C_2(s,N) &= -\frac{s}{9}\left[3D\pi^2s^{4/3}\left(2Ds^{2/3}+N^{5/6}\alpha\right)-8Nq^2D_1G_{10}\right] , \\
C_3(s,N) &= 3D^2N^{5/6}\pi^4s^{8/3}\alpha\left(6Ds^{2/3}-N^{5/6}\alpha\right)+256N^2q^4D_1^2G_{10}^2 \\
&\quad -24DN\pi^2q^2s^{4/3}D_1G_{10}\left(8Ds^{2/3}-N^{5/6}\alpha\right) .
\end{aligned}
$$



(a) $N = 5$

(b) $N = 10$

FIG. 9: The special heat $C_{N^2,q}$ as a function of s for the different numbers of colors with parameters $k = 1$, $q = 0.4$ and $\ell_p = 1$. Not that in this case, the critical value of the number of colors is $N_{crit} = \frac{12}{\alpha^2}\big(\frac{75D^2q^4D_1^2G_{10}^2}{\pi^4}\big)^{1/3} \approx 7.805$, so the specific heat has no divergent point in (a) for the case of $N = 5$, and two singularities at $s_3 \approx 1.167$ and $s_4 \approx 3.443$ in (b) for the case of $N = 10$.

Having considered the positiveness of the Hawking temperature (17), we find that $C_2(s, N)$ is always negative. The zero point of the specific heat is only relevant to the function $C_1(s, N)$, while the singularities of the specific heat are determined by the function $C_3(s, N)$. We show the behavior of $C_{\mu,q}$ as a function of s in Fig. (10). An interesting observation from the figure is that $C_{\mu,q}$ has a chance to be zero even for a non-extremal black hole. But the implication of this is not yet clear.

In the fixed q case, the Weinhold metric is two-dimensional and can be expressed as

$$g^W = \begin{pmatrix} \rho_{ss} & \rho_{sN^2} \\ \rho_{N^2s} & \rho_{N^2N^2} \end{pmatrix} . \tag{38}$$

The corresponding scalar curvature reads

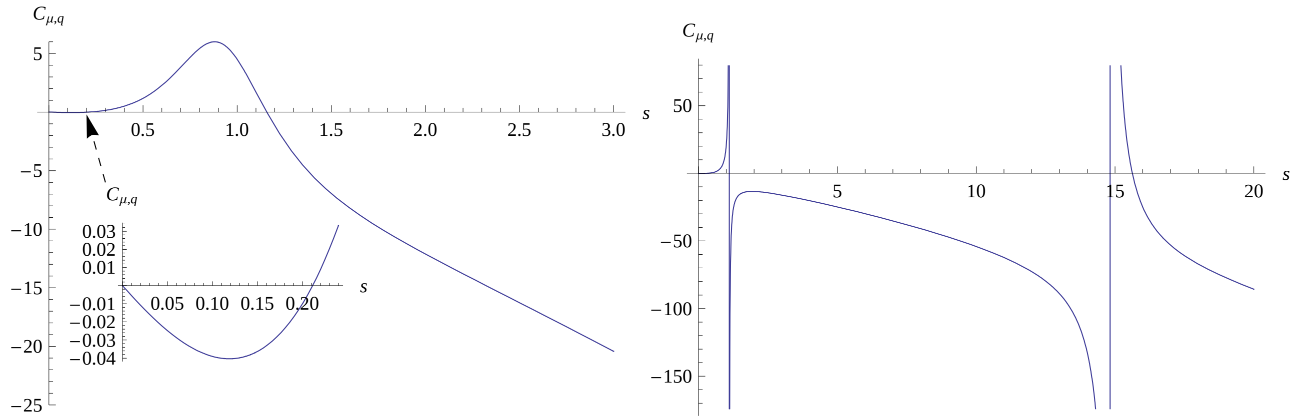
$$R^W = A_3(s, N)/B_3(s, N) , \tag{39}$$

where $A_3(s, N)$ is a complicated regular function which we do not present here, while

$$B_3(s, N) = N^{-3/2}[C_3(s, N)]^2 . \tag{40}$$

And the scalar curvature of Ruppeiner metric can also be obtained as

$$R^R = \frac{A_4(s, N)}{B_4(s, N)}, \tag{41}$$



(a) $N = 10$

(b) $N = 100$

FIG. 10: The specific heat $C_{\mu,q}$ vs entropy density s with parameters $k = 1$, $q = 0.4$ and $\ell_p = 1$. (a): The special heat has no divergence in the case of $N = 10$ but with zero point at $s_6 \approx 1.158$ (note that the other zero point at $s \approx 0.211$ corresponds to a vanishing Hawking temperature). (b): The specific heat has two singularities at $s_7 \approx 1.106$ and $s_8 \approx 14.825$ and a physical zero point at $s \approx 15.624$ in the case of $N = 100$.

with

$$B_4(s, N) = 27C_2(s, N)[C_3(s, N)]^2, \tag{42}$$

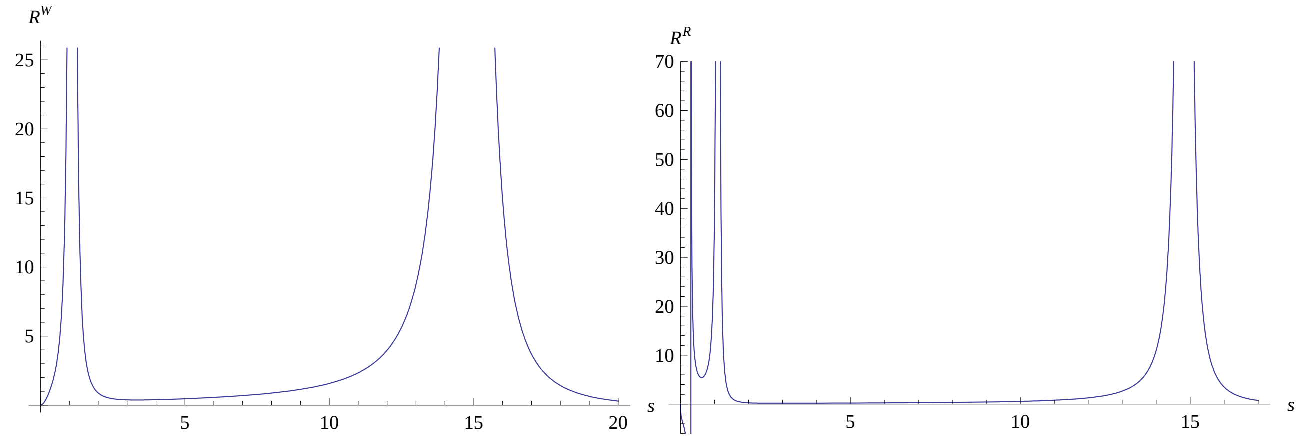
and again $A_4(s, N)$ is a regular function of s and N. In Fig. (11), we plot the scalar curvature for the Weinhold metric and Ruppeiner metric as a function of entropy density with parameters $q = 0.4$, $N = 100$ and $\ell_p = 1$. It is easy to see that both the Weinhold metric and Ruppeiner metric can give correct information of phase transition for the RN-AdS black hole in the fixed q case in the fixed chemical potential μ ensemble. Namely the singularity behavior of the thermodynamical curvature for both the Weinhold metric and Ruppeiner metric is consistent with that of $C_{\mu,q}$. Note that there is an additional singularity in plot (b) in Fig. (11), which corresponds to the extremal black hole with a vanishing Hawking temperature.

In the case with a fixed q, the Quevedo metric reads

$$g^Q = (sT + N^2\mu)\begin{pmatrix} -\rho_{ss} & 0 \\ 0 & \rho_{N^2N^2} \end{pmatrix} . \tag{43}$$

The corresponding scalar curvature can be expressed as

$$R^R = \frac{A_5(s, N)}{[C_1(s, N)]^2 E_5(s, N) B_5(s, N)} , \tag{44}$$



(a)Weinhold metric (b)Ruppeiner metric

FIG. 11: The scalar curvature of thermodynamical geometry vs entropy density for the (a) Weinhold metric and (b) Ruppeiner metric with parameters $k = 1$, $q = 0.4$, $N = 100$ and $\ell_p = 1$. Both the scalar curvatures diverge at $s_7 \approx 1.106$ and $s_8 \approx 14.825$, the same points of divergence of the specific heat $C_{\mu,q}$ (see plot (b) in Fig. (10)). Note that the other singularity in plot (b) corresponds to the extremal black hole with a vanishing Hawking temperature.

where $A_5(s,N)$ is a regular function, while the other two auxiliary functions are given by

$$E_5(s,N) = \left[3D\pi^2 s^{4/3}\left(4Ds^{2/3} + 3N^{5/6}\alpha\right) - 16Nq^2 D_1 G_{10}\right]^3 , \tag{45}$$

and

$$B_5(s,N) = \left[3D\pi^2 s^{4/3}\left(2Ds^{2/3} - N^{5/6}\alpha\right) + 40Nq^2 D_1 G_{10}\right]^2 . \tag{46}$$

With a positive Hawking temperature, the auxiliary function $E_5(s,N)$ is always positive. Then the divergence of the scalar curvature is related to the behaviors of $B_5(s,N)$ and $C_1(s,N)$. Comparing Eq. (44) with Eq. (27) and Eq. (37), we can conclude that the singularity behavior of the scalar curvature is the same as that of the specific heat of $C_{N^2,q}$, which means that the scalar curvature can reveal the information of phase transition of the black hole in the fixed N ensemble. In addition, let us note that there exists an additional singularity arising from the auxiliary function $C_1(s,N)$(e.g., $s_6 \approx 1.158$ in Fig. (12)), which corresponds to the zero point of the specific heat $C_{\mu,q}$. We can see these behaviors by comparing Fig. (12) with plot (b) in Fig. (9) and plot (a) in Fig. (10). These results are in agreement with the recent study in [35, 36] that the divergences of the scalar curvature for the Quevedo metric correspond to the singularities or zero for some specific heats.

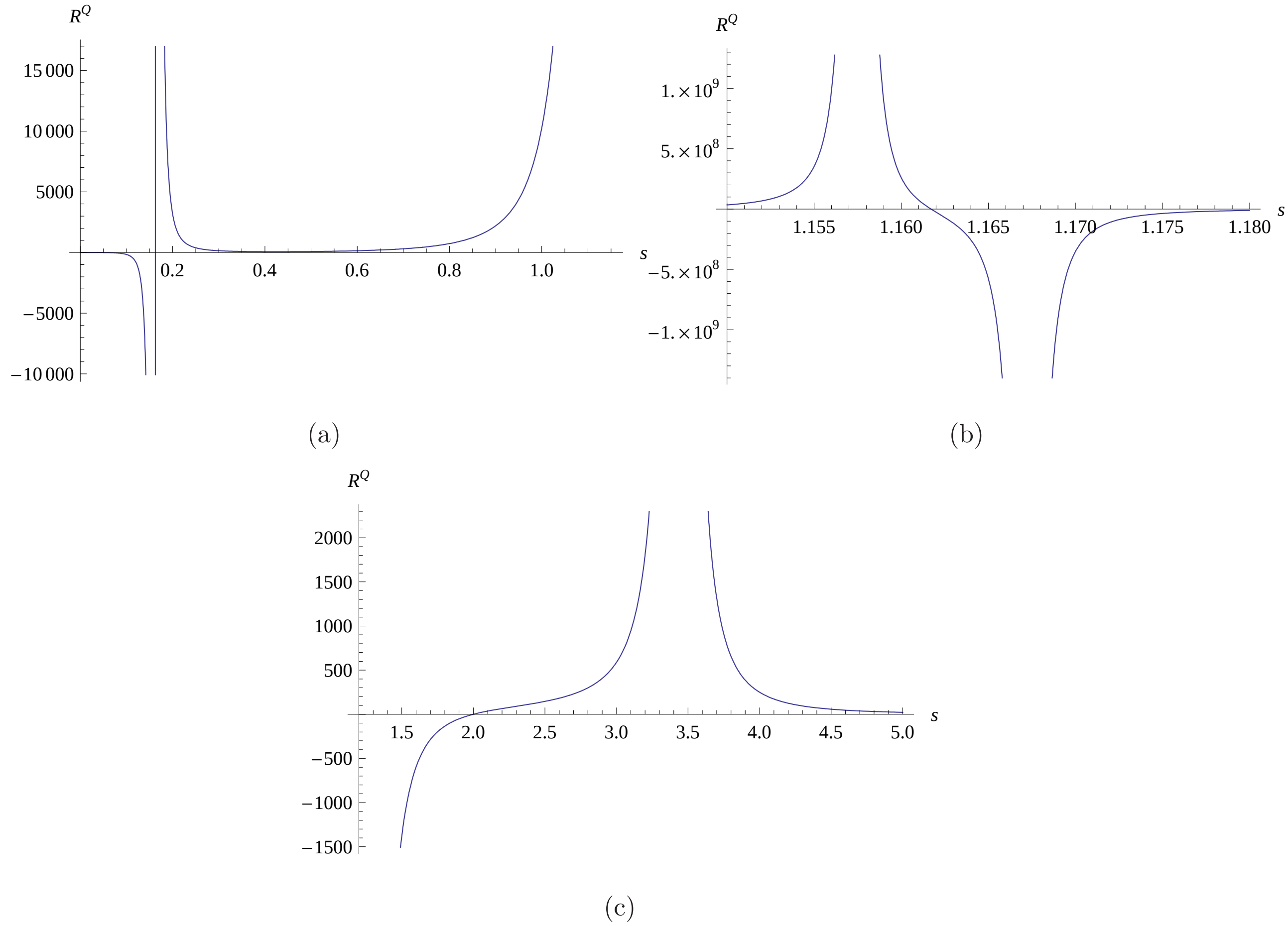


FIG. 12: The scalar curvature as a function of entropy density in the Quevedo metric with parameters $k = 1$, $q = 0.4$, $N = 10$ and $\ell_p = 1$. There exist three physical divergences at $s_6 \approx 1.158$, $s_3 \approx 1.167$ in plot (b) and $s_4 \approx 3.443$ in plot (c), respectively. And the remaining one at $s = 0.162$ in plot (a) corresponds a negative Hawking temperature, thus it is not physical relevant.

V. CONCLUSIONS

In this paper, we have studied the thermodynamics of a RN-AdS black hole in the extended phase space where the cosmological constant is related to the number of colors in the dual Supersymmetric Yang-Mills theory. Note that this extended phase space is different from the one where the cosmological constant acts as the pressure and its conjugate as the thermodynamical volume. In the former case, the mass of the black hole is viewed as the internal energy of the thermodynamical system, while in the latter case, the mass of the black hole acts as the enthalpy [11, 12]. We calculated and discussed the chemical potential associated with the number of colors, and found that the contribution of the charge of the

black hole to chemical potential is always positive. The chemical potential has a chance to be positive, and the existence of charge make the chemical potential become positive more easily. In the figure of the chemical potential as a function of temperature, we see that there exists a region where the chemical potential is a multi-valued function. This region just corresponds to the unstable branch of the black hole with a negative heat capacity. The chemical potential is found to be negative for large black holes, while it is positive for small ones. This behavior is qualitatively the same as that in the case of the Schwarzschild-AdS black holes [18].

The corresponding specific heats have been calculated respectively in the fixed N^2 case and the fixed q case. It has been found that the specific heat in the fixed number of colors and fixed charge density $C_{N^2,q}$ has a certain critical point, if the charge is larger than its critical value or the number of colors is smaller than its critical value, then $C_{N^2,q}$ has no divergence. The specific heat for a fixed electric potential and fixed number of colors C_{Φ,N^2} can not be zero for non-extremal black holes, while the specific heat for the fixed chemical potential and fixed charge $C_{\mu,q}$ has a chance to be zero even for a non-extremal black hole.

In the extended phase space, we have studied the thermodynamical geometry associated with the RN-AdS black hole. By calculating scalar curvatures of the Weinhold metric, Ruppeiner metric and Quevedo metric in the fixed N^2 case, we found that in the Weinhold metric and Ruppeiner metric both the scalar curvatures diverge at the same divergent points of C_{Φ,N^2}, while in the Quevedo metric, the scalar curvature diverges at the singularity of C_{q,N^2}. In the fixed q case, both the scalar curvatures in the Weinhold metric and Ruppeiner metric diverge at the divergent points of $C_{\mu,q}$, while the scalar curvature in the Quevedo metric diverges at the point of $C_{\mu,q} = 0$, besides the divergent points of $C_{N^2,q}$. These results indicate that the divergence of thermodynamical curvature indeed is related to some divergence of specific heats, but the divergence of thermodynamical curvature might be also relavent to the vanishing points of the specific heat. These studies are helpful to further understand the relation between phase transition and divergence of thermodynamical curvature. To further study this relation, it should be of great interest to discuss thermodynamics and thermodynamical curvature for other black holes in AdS space in the extended phase space. In particular, it is called for to further understand the potential role associated with the cosmological constant in the dual field theory according to the AdS/CFT correspondence.

Acknowledgments

R.G.C thanks Hunan Normal University for the warm hospitality extended to him during his visit. This work was supported in part by the National Natural Science Foundation of China (N0.11005038,No.10975168, No.11035008,No.11375092 and No.11435006), the Hunan Provincial Natural Science Foundation of China under Grant No. 11JJ7001 and the Program Excellent Talent Hunan Normal University No. ET13102.

---

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).

[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[4] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).

[5] O. Aharony, S.S. Guber, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rep. **323**, 183 (2000).

[6] S. W. Hawking and D. N. Page, Commun. Math.Phys.**87**, 577 (1983).

[7] P. C. W. Davies, Proc. Roy. Soc. Lond. A **353**, 499 (1977); P. C. W. Davies, Rep. Prog. Phys. **41**, 1313 (1978); P. C. W. Davies, Class. Quant. Grav. **6**, 1909 (1989).

[8] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. D **60**, 064018 (1999) [hep-th/9902170].

[9] J. y. Shen, R. G. Cai, B. Wang and R. K. Su, Int. J. Mod. Phys. A **22**, 11 (2007) [gr-qc/0512035].

[10] D. Kubiznak and R. B. Mann, JHEP **1207**, 033 (2012) [arXiv:1205.0559 [hep-th]].

[11] D. Kastor, S. Ray and J. Traschen, Class. Quant. Grav. **26**, 195011 (2009) [arXiv:0904.2765 [hep-th]].

[12] B. P. Dolan, Class. Quant. Grav. **28**, 235017 (2011) [arXiv:1106.6260 [gr-qc]].

[13] B. P. Dolan, arXiv:1408.4023 [gr-qc].

[14] S. Gunasekaran, R. B. Mann and D. Kubiznak, JHEP **1211**, 110 (2012) [arXiv:1208.6251 [hep-th]]; S. W. Wei and Y. X. Liu, Phys. Rev. D **87**, no. 4, 044014 (2013) [arXiv:1209.1707 [gr-qc]]; A. Belhaj, M. Chabab, H. El Moumni and M. B. Sedra, Chin. Phys. Lett. **29**, 100401 (2012) [arXiv:1210.4617 [hep-th]]; S. H. Hendi and M. H. Vahidinia, Phys. Rev. D **88**, no. 8, 084045

(2013) [arXiv:1212.6128 [hep-th]]; S. Chen, X. Liu, C. Liu and J. Jing, Chin. Phys. Lett. **30**, 060401 (2013) [arXiv:1301.3234 [gr-qc]]; E. Spallucci and A. Smailagic, Phys. Lett. B **723**, 436 (2013) [arXiv:1305.3379 [hep-th]]; R. Zhao, H. H. Zhao, M. S. Ma and L. C. Zhang, Eur. Phys. J. C **73**, 2645 (2013) [arXiv:1305.3725 [gr-qc]]; A. Belhaj, M. Chabab, H. El Moumni and M. B. Sedra, arXiv:1306.2518 [hep-th]; N. Altamirano, D. Kubiznak and R. B. Mann, Phys. Rev. D **88**, no. 10, 101502 (2013) [arXiv:1306.5756 [hep-th]]; R. G. Cai, L. M. Cao, L. Li and R. Q. Yang, JHEP **1309**, 005 (2013) [arXiv:1306.6233 [gr-qc]]; W. Xu, H. Xu and L. Zhao, Eur. Phys. J. C **74**, 2970 (2014) [arXiv:1311.3053 [gr-qc]]; J. X. Mo and W. B. Liu, Phys. Lett. B **727**, 336 (2013); D. C. Zou, S. J. Zhang and B. Wang, Phys. Rev. D **89**, 044002 (2014) [arXiv:1311.7299 [hep-th]]; J. X. Mo and W. B. Liu, Eur. Phys. J. C **74**, 2836 (2014) [arXiv:1401.0785 [gr-qc]]; N. Altamirano, D. Kubiznak, R. B. Mann and Z. Sherkatghanad, Galaxies **2**, 89 (2014) [arXiv:1401.2586 [hep-th]]; S. W. Wei and Y. X. Liu, Phys. Rev. D **90**, 044057 (2014) [arXiv:1402.2837 [hep-th]]; J. X. Mo, Europhys. Lett. **105**, 20003 (2014); J. X. Mo, G. Q. Li and W. B. Liu, Phys. Lett. B **730**, 111 (2014); J. X. Mo and W. B. Liu, Phys. Rev. D **89**, 084057 (2014) [arXiv:1404.3872 [gr-qc]]; D. C. Zou, Y. Liu and B. Wang, arXiv:1404.5194 [hep-th]; R. Zhao, M. Ma, H. Zhao and L. Zhang, Adv. High Energy Phys. **2014**, 124854 (2014); H. Xu, W. Xu and L. Zhao, arXiv:1405.4143 [gr-qc]; W. Xu and L. Zhao, Phys. Lett. B **736**, 214 (2014) [arXiv:1405.7665 [gr-qc]]; M. S. Ma and Y. Q. Ma, arXiv:1405.7609 [hep-th]; A. M. Frassino, D. Kubiznak, R. B. Mann and F. Simovic, arXiv:1406.7015 [hep-th]; G. Q. Li, Physics Letters B (2014), pp.256-260 [arXiv:1407.0011 [gr-qc]]; C. V. Johnson, arXiv:1406.4533 [hep-th]; C. O. Lee, arXiv:1408.2073 [hep-th].

[15] B. P. Dolan, arXiv:1406.7267 [hep-th].

[16] C. V. Johnson, arXiv:1404.5982 [hep-th];

[17] D. Kastor, S. Ray and J. Traschen, arXiv:1409.3521 [hep-th].

[18] J. L. Zhang, R. G. Cai and H. Yu, arXiv:1409.5305 [hep-th].

[19] F. Weinhold, J. Chem. Phys. **63**, 2479 (1975).

[20] G. Ruppeiner, Phys. Rev. A **20**, 1608 (1979).

[21] G. Ruppeiner, Rev. Mod. Phys. **67**, 605 (1995) [Erratum-ibid. **68**, 313 (1996)].

[22] P. Salamon, J. D. Nulton and E. Ihrig, J. Chem. Phys.**80**,436(1984).

[23] H. Quevedo, J. Math. Phys. **48**, 013506(2007).

[24] H. Quevedo, Gen. Relativ. Gravit. **40** , 971 (2008).

[25] H. Quevedo, A. Sanchez, S. Taj and A. Vazquez, Gen. Relativ. Gravit. **43**, 1153(2011).

[26] A. Bravetti, D. Momeni, R. Myrzakulov and H. Quevedo, Gen. Relativ. Gravit.**45**,1603(2013).

[27] S. Ferrara, G. W. Gibbons and R. Kallosh, Nucl. Phys. B **500**, 75 (1997) [hep-th/9702103].

[28] R. G. Cai and J. H. Cho, Phys. Rev. D **60**, 067502 (1999) [hep-th/9803261].

[29] J. E. Aman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. **35**, 1733 (2003) [gr-qc/0304015]; J. E. Aman and N. Pidokrajt, Phys. Rev. D **73**, 024017 (2006) [hep-th/0510139].

[30] G. Ruppeiner, "Breaking of Supersymmetry and Ultraviolet Divergences in Extended Supergravity," Springer Proceedings in Physics Volume 153, 2014, pp 179-203 [arXiv:1309.0901 [gr-qc]].

[31] S. A. H. Mansoori and B. Mirza, Eur. Phys. J. C **74**, 2681 (2014) [arXiv:1308.1543 [gr-qc]]; S. A. H. Mansoori, B. Mirza and M. Fazel,arXiv:1411.2582 [gr-qc].

[32] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **534**, 202 (1998) [hep-th/9805156].

[33] H.B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, second edition, 1985, John Wiley and Sons.

[34] C. Niu, Y. Tian, and X. Wu, Phys. Rev.D **85**, 024017(2012) [arXiv:1104.3066 [hep-th]].

[35] S. H. Hendi, Annals Phys. **346**, 42 (2014) [arXiv:1405.6996 [gr-qc]].

[36] J. Suresh, R. Tharanath, N. Varghese and V. C. Kuriakose, Eur. Phys. J. C **74**, 2819 (2014) [arXiv:1403.4710 [gr-qc]].